Chapter 2 – Circuit Elements

2.1 Introduction

$$Z = \frac{V}{I}$$

One of the three calculated parameters in electrical systems is impedance, which is the ratio of voltage to current. This is a restatement of Ohm's Law. In keeping with the Triad Principle, it would be expected that there are three elements of impedance.

The impedance elements are a resistor, inductor, and capacitor. A resistor is the property of any conductor of electrical energy. An inductor is comprised of a coil of wire. A capacitor exists between two adjacent conductors.

Impedance		Matter	Energy
Resistance	R	m	mechanical
Inductance	L	p	magnetic
Capacitance	C	q	electric

An electric energy field is associated with a capacitor. A magnetic energy field is associated with an inductor. Conversion to mechanical energy in the form of heat is associated with a resistor.

The three forms of energy are derived from the elements of matter- mass (m), charge (q), and magnetic poles (p). Mass yields mechanical energy, charge yields electric energy, and magnetic poles yield magnetic energy.

This chapter contains many of the items included in the Fundamentals of Engineering professional exam. This will be a compilation of the key areas associated with electric and magnetic storage devices and energy conversion element. The remainder of the book will focus on the interaction with the magnetic effects.

2.2 Circuit Element Construction

Impedance has only three linear (passive) elements available in a circuit: resistor, inductor, and capacitor. Resistors are a fixed property of direct current. Inductors and capacitors are a time varying, frequency property. They have duality with inductors as the positive mirror of the negative capacitance.

In physical systems, the elements are distributed over the length and area of a device. For example, transmission and power lines have the elements along the entire length and spread over the area of the conductor. However, in many problems, much simpler assumption is accurate enough. In this case the parameters are lumped at one location. The area and length are concentrated in a simple device that may be applied to a circuit or network.

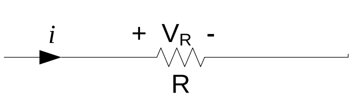
Note that for sinusoidal steady state conditions (SSS) there is no initial value or transient response.

$$s = j\omega$$

Similarly, for dc conditions, the frequency is zero.

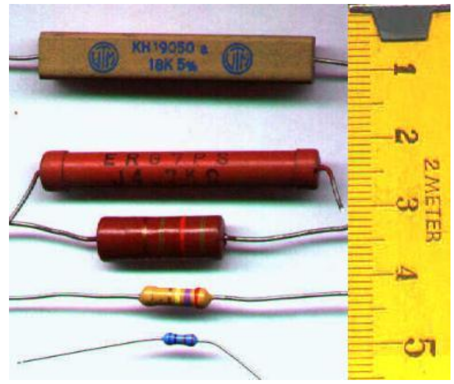
$$j\omega = 0$$

2.3 Resistor



A resistor converts electrical energy into mechanical energy in the form of heat. *Resistance* is measured in ohms (Ω). The voltage across a resistor, V_R, is the product of the resistance and the current through the resistor.

$$V_R = RI$$



A resistor is simply a length of conductive material. The resistance of a piece of material is given by

$$R = \rho_T \frac{l}{A}$$

Where

ρ_T - resistivity of construction material in Ohm-m

l is length of material, A is cross-sectional area

Material	ρ Ohm-m	α per °C
Copper	1.68×10^{-8}	.0068
Nichrome	1.10×10^{-6}	.0004
Silver	1.59×10^{-8}	.0061
Gold	2.44×10^{-8}	.0034
Aluminum	2.65×10^{-8}	.0043
Carbon	3.5×10^{-5}	-.0005
Germanium	4.6×10^{-1}	-.048
Silicon	6.40×10^{2}	-.075
Glass	10^{10} to 10^{14}	nil

The resistivity of the construction material, ρ_T, is temperature dependent. The resistance at a new temperature can be calculated from the resistivity at a reference temperature, ρ_T, using a temperature coefficient of resistivity, α. The reference temperature is generally 20C. A positive coefficient of resistivity indicates the resistance increases with temperature. The coefficient has units of per degree Celsius. At 21°C (294.15K), the resistivity of silver is 1.5938 x 10^{-8}.

$$\rho_T = \rho_{20}\left[1 + \alpha_{20}\left(T - 20\right)\right]$$

A color code is used to describe the quantity of resistance. Three or more bands are use. The first band gives the most significant digit. The second band has the next significant digit. The third band is the power of ten multipler. An optional fourth band gives tolerance while a fifth band provides a temperature coefficient.

Color	B1	B2	Mult	Tol	Temp
Black	0	0	10^0		
Brown	1	1	10^1	±1% (F)	100 ppm
Red	2	2	10^2	±2% (G)	50 ppm
Orange	3	3	10^3		15 ppm
Yellow	4	4	10^4		25 ppm
Green	5	5	10^5	±0.5% (D)	
Blue	6	6	10^6	±0.25% (C)	
Violet	7	7	10^7	±0.1% (B)	
Gray	8	8	10^8	±0.05% (A)	
White	9	9	10^9		
Gold			0.1	±5% (J)	
Silver			0.001	±10% (K)	
None				±20% (M)	

Resistors convert electrical energy into mechanical energy in the form of work and heat loss in the conversion. The conversion occurs as a power loss.

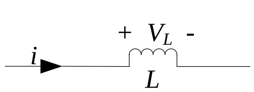
$$P = vi$$
$$= I^2 R$$
$$= \frac{V^2}{R}$$

Resistance is one of the three components of impedance.

$$Z_R = R$$

	EXAMPLES
Ex 2.3-1	Given: A cylinder of radius 1 cm, length of 2 cm, and made of germanium. Find: Resistance $R = \rho_T \frac{l}{A}$ $= \frac{\left(4.6\times10^{-1}\right)\left(2\times10^{-2}\right)}{\pi\left(1\times10^{-2}\right)^2} = 29.2\Omega$
Ex 2.3-2	Given: A copper wire sized #18 AWG with resistance of 7.77 Ohm/kfeet. Find: Resistance at 40C $\rho_T = \rho_{20}\left[1+\alpha_{20}\left(T-20\right)\right]$ $R = 7.77\left[1+.0068\left(40-20\right)\right] = 8.83\Omega / kft$
Ex 2.3-3	Given: A 60W lamp operating at 120 V. Find: Resistance $P = \frac{V^2}{R}$ $R = \frac{V^2}{P} = \frac{120^2}{60} = 240\Omega$
Ex 2.3-4	Given: R = 10 Ω. Find: Impedance at dc, 60 Hz, 1 MHz. $Z = R = 10\Omega$ R is independent of frequency.
Ex 2.3-5	Given a resistor has color bands of red, black brown. Find the resistance value. Red = 2 Black = 0 Brown = x10 Value = 2 – 0 – x10 = 200Ω

2.4 Inductor



An inductor converts electrical energy into magnetic energy. The inductor stores energy in a magnetic field. The *inductance* is measured in Henry (H). Because of the magnetic fields, the current through an inductor cannot change instantly. Rather, the current changes over time. Voltage across an inductor is the product of the inductance and the change of the current through the inductor over time.

$$V_L = L\frac{di}{dt}$$

The energy stored in an inductor comes from

$$dW = Vdq$$

$$W_L = \tfrac{1}{2}LI^2$$

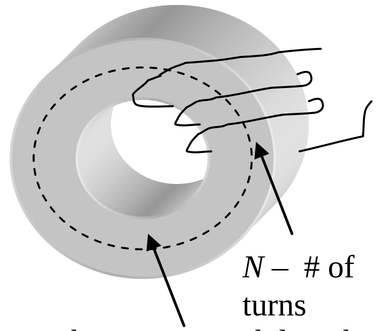
At its most basic form, an inductor is a coil of wire wrapped around a closed magnetic path. The inductance created is calculated from

$$L = \mu \frac{N^2 A}{l} = \frac{N^2}{\mathcal{R}}$$

Where μ – permeability of magnetic path in Henry/m

$$\mu = \mu_r \mu_o$$

$$\mu_o = 4\pi \times 10^{-7} \text{Henry/m}$$



Material	μ_r
Copper	≈2
Amorphous steel	≈2,000
Laminated steel	≈6,000

A conductor is a short circuit to direct current (dc). An inductor is a conductor for alternating current (ac) and the opposition to changes is proportional to frequency and inversely proportional to the time change. The first equation is the s-domain. The second is has the limitaion of sinusoidal steady state conditions.

$$Z_L = sL$$
$$= j\omega L$$

EXAMPLES

Ex 2.4-1

Given: An inductor uses 55 Watt-sec at a current of 15A.
Find: L

$$W_L = \tfrac{1}{2} L I^2$$

$$L = \frac{2W}{I^2} = \frac{2(55)}{15^2} = 0.49 Hy$$

Ex 2.4-2

Given: An inductor of 10 mHy has 100 turns and a rectangular length of 20 cm.
Find: Area of the coil for air and for laminated steel.

$$L = \mu \frac{N^2 A}{l}$$

$$A = \frac{Ll}{\mu N^2}$$

$$= \frac{(10\times10^{-3})(20\times10^{-2})}{\left[(4\pi\times10^{-7})(1)\right]\text{Henry/m}(100)^2} = 0.16 m^2$$

$$= \frac{(10\times10^{-3})(20\times10^{-2})}{\left[(4\pi\times10^{-7})(6000)\right]\text{Henry/m}(100)^2} = 2.7\times10^{-5} m^2$$

Ex 2.4-3

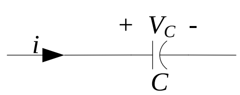
Given: An inductor of 10 mHy has 100 turns.
Find: Reluctance.

$$L = \frac{N^2}{\mathcal{R}}$$

$$\mathcal{R} = \frac{N^2}{L} = \frac{100^2}{10 mHy} = 10^6 turns^2 / Hy$$

Ex 2.4-4	Given: L= 10 mHy. Find: Z at dc, 60 Hz, and 1 Mhz. $Z_L = j\omega L$ $= j2\pi(0)10\times10^{-3} = 0$ $= j2\pi(60)10\times10^{-3} = j3.77$ $= j2\pi(10^6)10\times10^{-3} = j62{,}800$

2.5 Capacitor

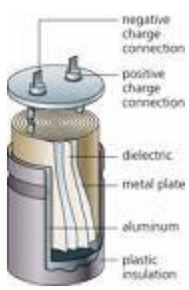


A capacitor stores electrical energy in an electric field. The *capacitance* is measured in Farads (F). The voltage across a capacitor is

$$v_c = \frac{1}{C}q = \frac{1}{C}\int i\,dt$$

Because of the electric field, the voltage across a capacitor (v_c) cannot change instantaneously. The current through a capacitor is found from the change in voltage over time.

$$i = C\frac{dv_c}{dt}$$

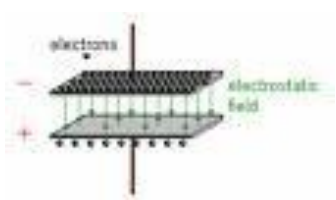


The amount of energy stored in a capacitor is shown.

$$dW = Vdq = CVdv$$

$$W_C = \tfrac{1}{2}CV_c^2 = \tfrac{1}{2}\frac{q^2}{C}$$

A capacitor is constructed of two conductors, separated by some dielectric material. The capacitance generated is a product of the permittivity and the size of the conductors.



$$C = \varepsilon\frac{A}{l}$$

Where

A – cross-sectional area of conductors

l – separation of conductors

ε – permittivity of dielectric material in Farad/m.

Material	ε_r
Glass	5 - 10
Germanium	16
Titanium dioxide	86 - 173

The permittivity of the dielectric material is referenced to the permittivity of free space

$$\varepsilon = \varepsilon_o\varepsilon_r$$

Where

ε_o - permittivity of free space - $\dfrac{1}{36\pi\times10^9}$ Farad/m

ε_r - dielectric constant (relative permittivity)

The dimensions of the cross-sectional area are very large. Capacitors typically consist of two pieces of foil separated by a dielectric. The terminals are attached to each of the layers of foil. The capacitor foil and dielectric is then rolled in a cylinder. This provides a large surface area, but it is contained in a relatively small volume.

It would be impractical to use an air dielectric capacitor for most circuits. However, when wires are adjacent to another conductor or the earth, a capacitance is developed. For long conductors, this distributed capacitance can have a substantial impact.

A capacitor is an open circuit to direct current (dc). However, a dc voltage will charge a capacitor to the level of the voltage. A capacitor is a conductor for alternating current (ac) and the opposition to changes is proportional to the time of change and inversely proportional to the frequency. The first equation is in the s-domain. The second has the constraint of sinusoidal steady state conditions.

$$Z_C = \frac{1}{sC} = \frac{1}{j\omega C}$$

$$c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

$$U = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\frac{1}{\sqrt{\mu_r\varepsilon_r}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}}$$

Velocity is inversely proportional to the square root of permeability and permittivity

EXAMPLES

Ex 2.5-1

Given: A capacitor of 10 µFd at 12 Vdc.
Find: Energy stored and the charge.

$$W_C = \tfrac{1}{2} C V_c^2 = \tfrac{1}{2}\left(10\times10^{-6}\right)12^2 = 0.72mJ$$

$$W = \tfrac{1}{2}\frac{q^2}{C}$$

$$q = \sqrt{2WC} = \sqrt{2\left(0.72mJ\right)\left(10\times10^{-6}\right)} = 120\mu C$$

Ex 2.5-2

Given: A capacitor of 10 µFd has 1 mm between the plates.
Find: Area of the capacitor using air and titanium dioxide as the dielectric.

$$C = \varepsilon\frac{A}{l}$$

$$A = \frac{Cl}{\varepsilon} = \frac{\left(10\times10^{-6}\right)\left(1\times10^{-3}\right)}{\left(\frac{1}{36\pi\times10^9}\right)(1)} = 1131m^2$$

$$= \frac{\left(10\times10^{-6}\right)\left(1\times10^{-3}\right)}{\left(\frac{1}{36\pi\times10^9}\right)(100)} = 11.31m^2$$

Ex 2.5-3

Given: 10 µFd.
Find: impedance at dc, 60 Hz, 1 MHz.

$$Z_C = \frac{1}{j\omega C}$$

$$= \frac{1}{j2\pi(0)(10\times10^{-6})} = -j\infty$$

$$= \frac{1}{j2\pi(60)(10\times10^{-6})} = -j265\Omega$$

$$= \frac{1}{j2\pi(10^6)(10\times10^{-6})} = -j0.0159$$

2.6 Element Basics

2.6.1 Element Reciprocals

$Z = R + jX$

Because of circuit combinations, reciprocal values of elements often arise. Where impedance (Z) is opposition to current flow, the reciprocals of impedance indicate the ease of current flow. Impedance is made up of two components, the resistance (real portion) and the reactance (imaginary portion), which is the combination of inductance and capacitance.

Parameter	Reciprocal of	Symbol	Relation	Unit	Symbol
Impedance		Z		Ohm	Ω
Admittance	Impedance	Y	$Y = \frac{1}{Z}$	mho	
Conductance	Resistance	G	$G = \frac{1}{R}$	mho	
Susceptance	Reactance	G_B	$G_B = \frac{1}{X}$	Siemen	S
Elastance	Capacitance	E_C	$E_C = \frac{1}{C}$		
Reluctance	Inductance	$\Re$	$\Re = \frac{N^2}{L}$	Amp-turn / Weber	

Impedance is the sum of resistance and reactance *j:

$$Z = R + jX$$

Admittance is the sum of conductance and susceptance:

$$Y = G + jG_B$$

2.6.2 Reactance

Reactance is the imaginary, time varying, frequency component of impedance. The first equation is in the s-domain. The second is constrained to sinusoidal steady state conditions.

$$jX = sL + \frac{1}{sC}$$
$$= j\omega L + \frac{1}{j\omega C}$$

2.6.3 Element Combinations

The elements defined in the previous topics can be connected into combinations of impedances. The configuration of the combinations determines the mathematical treatment of the impedances (Z).

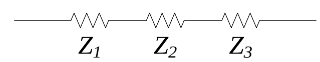
Impedance is the ratio of voltage to current. Voltage and current may be represented in time or by one of many transforms. For impedance combinations, the transform is generally applied to the impedance. That technique recognizes the common energy in the impedance path.

The bases of all combinations are the series and parallel connections.

2.7 Series

In a series circuits, the current will be the same in all elements, so the impedance is proportional to the voltage. *Series* impedances are *added.*

$$Z_{total} = Z_1 + Z_2 + Z_3 \ldots$$



2.7.1 Resistors

For resistors, the impedance is equal to the resistance, $Z_R = R$.

$$Z_{R-total} = R_1 + R_2 + R_3 \ldots$$

2.7.2 Inductors

The impedance of inductors is frequency dependant and is proportional to the inductance, $Z_L \propto L$. Series inductors at the same frequency are treated the same as resistances.

$$Z_{L-total} = s(L_1 + L_2 + L_3)$$
$$= j\omega(L_1 + L_2 + L_3)$$

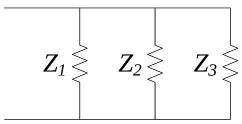
2.7.3 Capacitors

Capacitors are a different animal. The impedance of capacitors is frequency dependant, but is inversely proportional to the capacitance, $Z_C \propto \frac{1}{C}$. For this reason, when capacitors are in series, the series capacitance is the sum of the elastance, or $\frac{1}{C}$.

$$Z_{C-total} = \frac{1}{s}\left(\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}\right)$$
$$= \frac{1}{j\omega}\left(\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}\right)$$

2.8 Parallel

In a parallel circuit the voltage is the same across all elements, so the impedance is proportional to the current. Parallel impedances are the sum of the reciprocals. The reciprocal of the impedance is the admittance of each element. This is the reciprocal of the impedance of the circuit.



$$\frac{1}{Z_{total}} = Y_{total} = \frac{1}{Z_1}+\frac{1}{Z_2}+\frac{1}{Z_3}$$

The special case of two parallel impedances can be found by taking the product of the impedances and dividing by the sum of the impedances.

$$Z_{total} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

Similarly the three impedances include the combinations of two impedances.

$$Z_{total} = \frac{Z_1Z_2 + Z_1Z_3 + Z_2Z_3}{Z_1 + Z_2 + Z_3}$$

2.8.1 Resistors

For resistors, the admittance of the circuit is the sum of the conductance of the individual elements.

$$\frac{1}{Z_{R-total}} = G_1 + G_2 + G_3 \ldots = \frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\ldots$$

2.8.2 Inductors

For inductors, the admittance of the circuit is proportional to the sum of the reluctance of the individual inductors.

$$\frac{1}{Z_{L-total}} = \frac{1}{s}\left(\frac{1}{L_1}+\frac{1}{L_2}+\frac{1}{L_3}\right)$$
$$= \frac{1}{j\omega}\left(\frac{1}{L_1}+\frac{1}{L_2}+\frac{1}{L_3}\right)$$

2.8.1 Capacitors

Again, capacitors are a special case. For capacitors at the same frequency, the admittance is proportional to the sum of the capacitances.

$$\frac{1}{Z_{C-total}} = s\left(C_1 + C_2 + C_3\right)$$
$$= j\omega\left(C_1 + C_2 + C_3\right)$$

	EXAMPLES
Ex 2.8-1	Given: Three resistors of 10 Ω. Find: Series resistance. $Z_R = R_1 + R_2 + R_3 \ldots$ $Z = 3(10) = 30\Omega$
Ex 2.8-2	Given: Three resistors of 10 Ω. Find: Parallel resistance. $\frac{1}{Z_R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \ldots$ $\frac{1}{Z_R} = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = 0.3$ $Z_R = 3.33\Omega$ An alternate solution provides additional insight. $Z_{total} = \frac{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}{Z_1 + Z_2 + Z_3}$ $Z_{total} = \frac{(10)(10) + (10)(10) + (10)(10)}{10 + 10 + 10} = 3.33\Omega$
Ex 2.8-3	Given: Two capacitors of 10 μFd. Find: Series capacitance. $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$ $= \frac{1}{10\times10^{-6}} + \frac{1}{10\times10^{-6}} = 0.2\times10^{6}$ $C = \frac{\left(10\times10^{-6}\right)\left(10\times10^{-6}\right)}{2\left(10\times10^{-6}\right)} = 5\mu Fd$
Ex 2.8-4	Given: Two capacitors of 10 μFd. Find: Series impedance operating at 1 MHz. $Z_C = \frac{1}{j\omega}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$ $= \frac{1}{j2\pi10^{6}}\left(\frac{1}{10\times10^{-6}} + \frac{1}{10\times10^{-6}}\right) = 0.0318\Omega$ $Z = \frac{1}{j2\pi10^{6}\left(5\mu Fd\right)} = 0.0318\Omega$
Ex 2.8-4	Given: Two capacitors of 10 μFd. Find: Parallel capacitance. $C = \left(C_1 + C_2\right)$ $= 10\times10^{-6} + 10\times10^{-6} = 20\mu Fd$

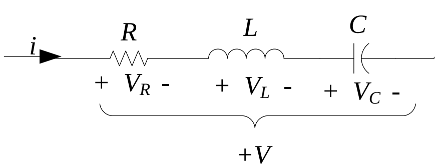
Ex 2.8-6	Given: Two capacitors of 10 μFd. Find: Parallel impedance operating at 1 MHz.

$$\frac{1}{Z_{C-total}} = j\omega\left(C_1 + C_2 + C_3\right)$$

$$Z_C = \frac{1}{j\omega\left(C_1 + C_2\right)}$$

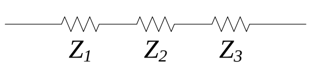
$$= \frac{1}{j2\pi\left(10^6\right)\left(10\times10^{-6} + 10\times10^{-6}\right)} = 0.00795\Omega$$

2.9 Impedance Combinations



The three elements of impedance are resistor, inductor, and capacitor. Each is unique in its relationship with time and frequency. Therefore, the combinations are unique. All three elements existing together form the most complex configuration in physical systems. These combinations can ultimately be reduced to a series or parallel network.

2.9.1 Series

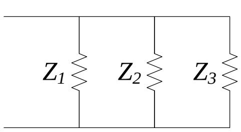


The series combination of impedance is simply an addition of the individual impedance elements. The relationships are shown in both the s-domain and the sinusoidal steady state condition.

$$Z = R + sL + \frac{1}{sC}$$

$$= R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

2.9.2 Parallel



The parallel combination of impedance is reciprocal of the sum of the reciprocals of the individual impedance elements. The relationships are shown in both the s-domain and the sinusoidal steady state condition.

$$\frac{1}{Z} = Y = \frac{1}{R} + \frac{1}{sL} + sC$$

$$= \frac{1}{R} + \frac{1}{j\omega L} + j\omega C$$

2.9.3 Reciprocal

The time varying or frequency components are defined as reactance.

$$jX = sL + \frac{1}{sC}$$

$$= j\omega L + \frac{1}{j\omega C}$$

A common problem is addressing the combination of reciprocals, particularly as associated with parallel combinations. The computations require complex math. The relationships are illustrated.

$$Y = \frac{1}{R} + \frac{1}{jX} = G - jG_B$$
$$= \frac{1}{R} - \frac{j}{X}$$
$$= \frac{R + jX}{jXR}$$

When converting to impedance, simply take the reciprocal of the real and the imaginary.

$$Z = \frac{jXR}{R + jX}$$
$$= \frac{1}{G} - \frac{1}{jG_B}$$

The impedance result is the familiar form for two parallel impedances. It is the product over the sum of the individual elements. Clearly the definitions of reactance for the inductor and capacitor can be substituted into the equations to give an s-domain for SSS form.

	EXAMPLES
Ex 2.9-1	Given: R = 10 Ω, L = 10 mHy, and C = 10 µFd. Find: Series Z at 1 kHz. $Z = R + j\omega L + \frac{1}{j\omega C}$ $= 10 + j\left(2\pi(1000)(10\times10^{-3}) - \frac{1}{2\pi(1000)(10\times10^{-6})}\right)$ $= 10 + j\left(20\pi - \frac{1}{20\pi\times10^{-3}}\right) = 10 + j46.9\Omega$
Ex 2.9-2	Given: R = 10 Ω, L = 10 mHy, and C = 10 µFd. Find: Parallel Y at 1 kHz. $Y = \frac{1}{R} + \frac{1}{j\omega L} + j\omega C$ $= \frac{1}{10} + \frac{-j}{2\pi(1000)(10\times10^{-3})} + j2\pi(1000)(10\times10^{-6})$ $= 0.1 + j\left(20\pi - \frac{1}{20\pi\times10^{-3}}\right) = 0.10 + j46.9\Omega$

Ex 2.9-3	Given: R = 10 Ω, L = 10 mHy, and C = 10 μFd. Find: Parallel Z at 1 kHz.

$$Z = \frac{1}{Y} = \frac{1}{G} + \frac{1}{G_B}$$

$$= \frac{1}{0.10} + \frac{1}{j46.92} = 10 - j0.02\Omega$$