Chapter 3 - Methods

3.1 Introduction

One of the first requirements for analysis is mathematical tools, techniques, and transforms. Next a fundamental understanding of electrical measurements and calculations is necessary. Then the construction of passive elements to define impedance can be accomplished. Now, circuit performance can be analyzed. Numerous methods are available. These practices are developed from the electromagnetic energy equation and the conservation of energy [1].

$$W = \frac{pq}{t}$$

$$\Sigma W = 0$$

3.2 Electrical laws

The concepts embedded in the very fundamental node form of the electromagnetic energy equation are staggering.

3.2.1 Definitions

First, the definition of voltage, current, and frequency is contained in the expression.

How many more relationships can be found? Consider just a few. Electrical energy is voltage multiplied by charge. Magnetic energy is current multiplied by magnetic pole flux. These relationships were used to describe the energy storage in impedance elements.

$$W_{Electric} = vq$$

$$W_{Magnetic} = ip$$

3.2.2 Kirchhoff

Now the plot thickens even further. Circuit analysis is often described using two laws developed by the Prussian mathematician and physicist, Gustav Robert Kirchhoff in 1854.

Both these laws are imbedded in the very simple electromagnetic energy definition. First, apply the constraint of conservation to the relationship. This sets the sum of the energy equal to zero. Next, hold one term constant. Then, the sum of the changing term is zero.

Kirchhoff’s current law (KCL) can be stated succinctly:

When the magnetic flux is constant, the sum of the current at a node is zero.

$$\Sigma W = 0$$

$$\Sigma \frac{pq}{t} = 0$$

$$\Sigma I = 0\Big|_{p=k}$$

Similarly, Kirchhoff’s voltage law (KVL) can be stated:

When the charge is constant, the sum of the voltage around a loop is zero.

$$\Sigma \frac{pq}{t} = 0$$

$$\Sigma V = 0\Big|_{q=k}$$

3.2.3 Faraday

Faraday's law is also imbedded in the electromagnetic energy correlation. It states that the rate of change of the magnetic flux or pole strength is equal to the induced voltage. This is the definition of voltage.

$$V_{induced} = \frac{p}{t}\Big|_{q=k}$$

3.2.4 Conservation

When conservation of energy is applied to the electromagnetic energy expression, an entire paradigm is developed. Since charge, flux, and time cannot convert to the other, conservation applies to each item individually.

$$\Sigma W = 0$$

$$\Sigma \frac{pq}{t} = 0$$

Conservation of charge states the sum of the charge is zero. Charge is discrete and is an integral multiple of the charge on an electron or proton.

$$q = 1.60217646 \times 10^{-19} \text{ Coulombs}$$

The total charge is always balanced. A proton is 1836 times heavier than an electron. Nevertheless, it has exactly the same charge.

$$\Sigma q\Big|_{\frac{p}{t}=k} = 0$$

$$\Sigma p\Big|_{\frac{q}{t}=k} = 0$$

$$\Sigma \frac{1}{t}\Big|_{pq=k} = 0$$

Conservation of magnetic pole strength states the sum of the magnetic flux is zero. Magnetic poles always exist in a balanced pair with a north and south poles.

Conservation of time and frequency implies that she sum of frequency is zero. Alternatively, time is conserved. Conservation of time and frequency is directly related to Planck. The number of waves or cycles, w, is a discrete number, just like charge, q.

$$W = h\frac{w}{t}$$

$$h = \text{Planck's constant}$$

$$\text{w=descrete numbers}$$

These relationships are implicit in the electromagnetic expression when the constraint of conservation of energy is imposed.

Spectrum	Frequency
Direct current	0
AC power	60 Hz
Sound	1 kHz
AM radio	1MHz
FM radio	100 MHz
UHF TV	500 MHz
Cell phone 1	800 MHz
Cell phone 2	1.9 GHz
Satellite radio	2.2 GHz
Wireless LAN	2.4 GHz
Microwave	2.5 GHz
Radar	5.0 GHz
Infrared	
Visible	
Ultraviolet	
X-ray	
Gamma ray	
All objects have an electro-magnetic (radio) frequency.	

3.2.5 Power

Power is the energy over time. Power imposes another time on the electromagnetic energy.

$$p = \frac{W}{t_r}$$

$$= \frac{pq}{t_r t_t}$$

$$= vi$$

3.2.6 Complete

Other than Ohm's Law, which provides the definition of impedance, all electrical laws and relationships are contained in the electromagnetic energy relationship.

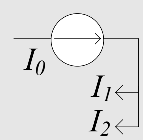
$$Z = \frac{V}{I}$$

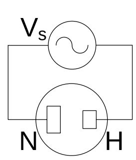
The complete suite of relationships used for circuit analysis is rooted in one, very simple, elegant equation.

The development and application of the concepts can most often be done with mathematics no more complex than algebra. This opens the understanding of electromagnetic science to an entire new level of application.

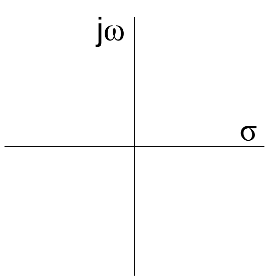
3.2.7 Fields

Another class of analysis embraces electromagnetic energy dispersed through a medium, rather than focused at a point or node. The investigation requires knowing the location in space. Expansion to the field form of electromagnetic energy includes distances and directions.

	EXAMPLES
Ex 3.2-1	Situation: Figure Find: Use KCL to obtains the equation for I_2 $\Sigma I = 0$ $I_0 + I_1 + I_2 = 0$ $I_2 = I_0 - I_1$ 
Ex 3.2-2	Situation: I_0=10, I_1=2 Find I_2 $I_2 = 10 - 2$ $I_2 = 8A$
Ex 3.2-2	Situation: Figure Find: Use KVL to find the voltage between the H-N terminals. $\Sigma V = 0$ $V_N - V_S + V_H = 0$ $V_S = V_H - V_N = V_{HN}$



$s = \sigma + j\omega$

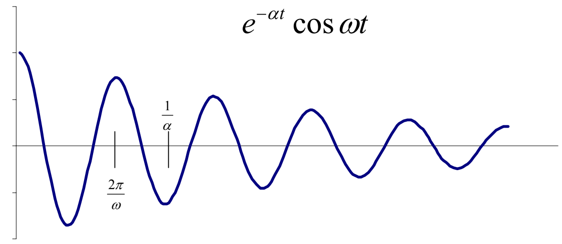


3.3 Laplace dc

The Laplace transform is defined in terms of a real, stability factor and the orthogonal frequency. In a large class of problems, the frequency is zero. This is a direct current or step function.

$$s = \sigma + j\omega$$

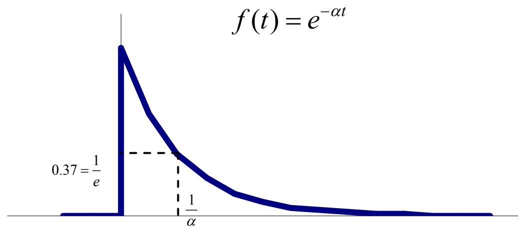
$$s = \sigma$$



Frequency is dependent on the storage elements, capacitors and inductors. If either of these is missing, then there is not a frequency component to the Laplace.

If there is only one storage element, then the response will be an exponential decay.

If there are no storage elements, then the impedance is only resistance. Then, the response will be a step function with the magnitude determined by the source and resistance.

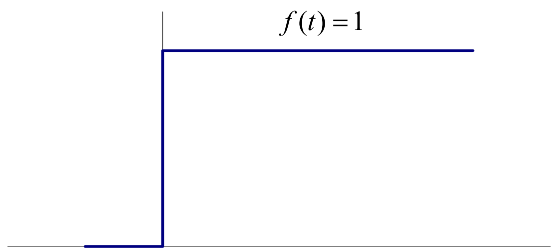


$$Z = R + sL + \frac{1}{sC}$$

Write Ohm's Law where current is the response and voltage is the source.

$$Z(s) = \frac{V(s)}{I(s)}$$

$$I(s) = \frac{V(s)}{Z(s)}$$



Consider a network with only resistance and a step or dc voltage supply or source. Find the current in Laplace form.

$$I(s) = \frac{V}{s}\left[\frac{1}{R}\right]$$

Now take the inverse Laplace to find the current in the time domain.

$$I(s) = \left[\frac{V}{R}\right]\frac{1}{s}$$

$$i(t) = \frac{V}{R}$$

For the special case of a resistance network and a direct current, step source, then the time domain response is simply the magnitudes of the Laplace coefficients. Therefore, there is no loss of accuracy to simply write the equations without using the step operator, *1/s,* and then finding the inverse Laplace.

3.4 Branches, Nodes, Loops

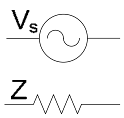
A circuit is defined as a network or combination of sources and impedances. The sources can be either voltage or current. The impedances can be resistors, inductors, and capacitors. In addition active elements consisting of electronic devices can be included.

3.4.1 Superposition

The unknowns can be voltage, current, or impedance at any location within the circuit. The law of superposition allows the investigation to be broken down into smaller pieces. The system is the sum of the behavior of the individual components acting separately.

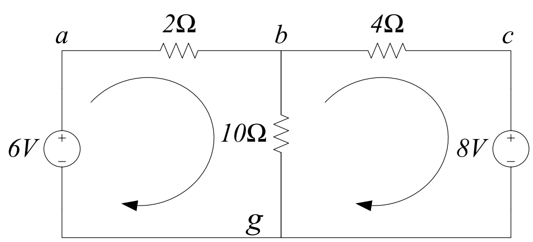
For the purposes of circuit analysis, all impedances will be drawn as resistors and labeled in Ohms. Conversion can be made from the impedance back to the resistors, inductors, and capacitors.

3.4.2 Topology definitions



A *branch* is a single element such as a source or an impedance. A branch is a two terminal element. The network figure has three impedances and two sources to give five branches.

A *node* is the point of connection of two or more branches. A single node results if a wire without impedance connects two nodes. The network figure has four nodes labeled a, b, c and g. The g node is a reference and is often connected to ground.



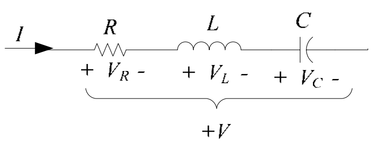
A *loop* is a closed path in a circuit. A loop starts at a node, passes through other nodes and returns to the starting node without passing through any node more than one time. A loop is sometimes called a mesh or a pane as in window.

An independent loop contains at least one branch that is not part of any other loop. The figure has two independent loops.

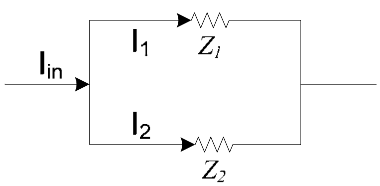
3.4.3 Topology Theorem

The theorem of network topology relates the components. The number of branches (b) is the sum of the loops (l) plus the nodes (n) minus 1.

$$b = l + n - 1$$

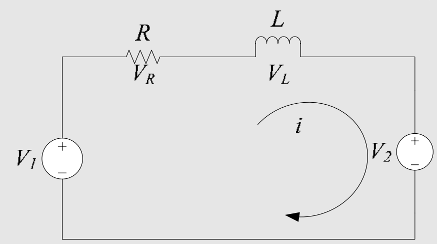


Series elements are connected so they exclusively share one node and carry the same current.



Parallel elements are connected to the same two nodes and have the same voltage across them.

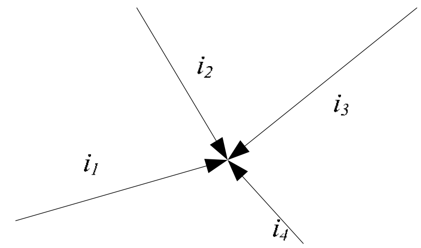
Other connections exist where more than two elements are connected to a node. The combinations require the application of more sophisticated techniques for analysis. The remainder of the chapter is devoted to these techniques.

	EXAMPLES
Ex 3.2-1	Situation: Figure Find: the number of branches, nodes, and loops.  Nodes=4 Loop = 1 Branches=2 sources + 2 impedance = 4 Check: b=l+n-1 4=1+4-1 OK

3.5 Kirchhoff laws

3.5.1 Circuit Laws

Two circuit laws are used in all circuit analysis. These circuit laws adhere to the Conservation of Energy – There is nothing new under the sun; or, more traditionally, the sum of the energy in a closed system is zero.

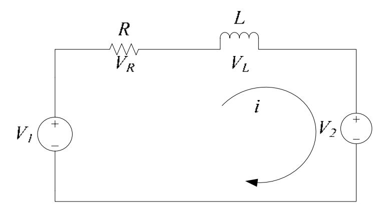


Kirchhoff's Current Law (KCL) relates to currents entering a node. KCL states that the sum of currents entering a node is equal to zero (0).

$$\Sigma i_n = 0$$

$$i_1 + i_2 + i_3 + i_4 = 0$$

By convention, if current enters a node, it is considered negative. Current leaving a node is considered positive.



Kirchhoff's Voltage Law (KVL) relates to voltages in a circuit loop. KVL states that the sum of voltages in a loop is equal to zero (0).

$$\Sigma v_n = 0$$

$$-v_1 + v_R + v_L + v_2 = 0$$

$$V_1 - V_2 = \left(R + sL\right) I$$

By convention, if current goes into "–" of voltage *source*, then the voltage is considered positive. If current goes into "+" of voltage source, then the voltage is considered negative. If current goes into the"+" of an *impedance*, then it is a voltage drop. If current goes into the "–" of an impedance, then it is a voltage rise.

3.5.2 Circuit Analysis

Circuit analysis is obviously a detailed process that requires many steps. Nevertheless, there is a general approaches that applies to any resolution of any circuit problem. Five rules aid in any circuit analysis.

1) Assume a consistent group of currents & voltages for each impedance element (R, L, C). Sketch these currents & voltages.
2) Conservation: Write equations using Kirchhoff's Law (KCL or KVL).
3) Substitute element definitions (Ohm's Law) for each impedance.
4) Combine equations in terms of unknowns, either current or voltage.
5) Solve simultaneous equations for unknown voltage or current using Cramer's rule or elimination.
6) If necessary, calculate any other desired value from the unknown.

Circuit analysis has a source (external forcing function for voltage or current) and impedance elements (opposition).

The "answer" to a circuit analysis problem is the voltage and current across an impedance element.

Occasionally the answer is a derived value such as power or energy. However, that requires the voltage and current. Once the current and voltage are known, then any impedance that is unknown can be found.

	EXAMPLES
Ex 3.5-1	Situation: The current diagram above with I_1=-1, I_2= 2,I_3=3. Find: I_4. $I_1 + I_2 + I_3 + I_4 = 0$ $I_4 = 1 - 2 - 3 = -4A$
Ex 3.5-2	Situation: Loop diagram above with R=10 Ohms and L=10 mHy. V_1=24 and V_2=12. Find: Voltage V_{12}. $V_{12} = V_1 - V_2$ $= 24 - 12 = 12V$
Ex 3.5-3	Find: I(s) $I(s) = \frac{V_{12}}{s}\frac{1}{sL+R}$ $= \frac{12}{s}\frac{1}{10^{-2}s+10}$ $= \frac{12\times10^2}{s\left(s+10^3\right)}$ Now expand into partial fractions $I(s) = \frac{k_f}{s} + \frac{k_1}{s+10^3}$ Multiply through and equate numerators. $12\times10^2 = k_f\left(s+10^3\right) + k_1 s$ $0 = \left(k_f + k_1\right)s$ $12\times10^2 = 10^3 k_f$ Determine coefficients. $k_f = 1.2$ $k_1 = -k_f = -1.2$ Write I(s) from the partial fraction expansion. $I(s) = \frac{1.2}{s} - \frac{1.2}{s+10^3}$
Ex 3.5-4	Find: i(t) $i(t) = 1.2 - 1.2e^{-1000t}$
Ex 3.5-5	Find: time constant. $t_c = \frac{1}{\alpha}$ $= \frac{1}{1000} = 10^{-3}\text{ sec}$
Ex 3.5-6	Find: Final value. $i(\infty) = 1.2$

3.6 Methods

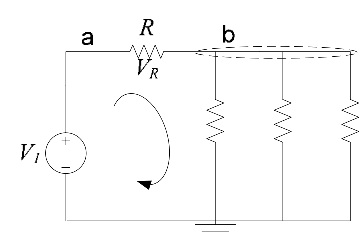
Three methods of solving circuit analysis are commonly used. These are directly related to and derived from Kirchhoff's Law.

1. Kirchhoff's Law
    a. Loop current uses KVL and is called mesh analysis.
    b. Node voltage uses KCL and is called nodal analysis.
2. Equivalent impedances are combinations of KVL and KCL to reduce the problem to simpler structure.
    a. Series / parallel combination
    b. Voltage / current dividers
    c. Delta-wye conversion
3. Equivalent source is based on Ohm's Law
    a. Thevenin equivalent voltage source
    b. Norton equivalent current source

3.6.1 Nodal - node voltage

The nodal method is a parallel approach to the problem. Nodal analysis has broad applications outside of electrical engineering, since it only requires information at specific locations or nodes and general information between the nodes. Nodal analysis has been adopted by petroleum engineers to describe the performance of underground reservoirs of oil and gas.

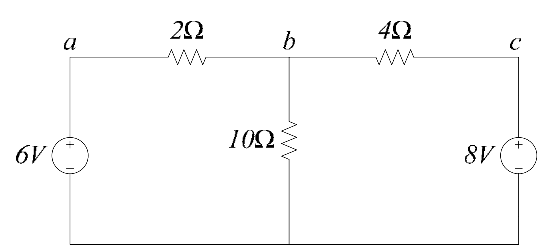
The application of the general rules for circuit analysis yield specific steps for nodal analysis.



1) Assign node voltage at each connection. (Ground = ref = 0V)
2) Assume a branch current direction (eg. always leaves node).
3) Write KCL at each node.
4) Substitute impedance elements (Ohm's law) for current.
5) Solve for unknown voltage at each node.
6) The impedances are known. Once the voltage is found at each node, then the current in each branch can be determined.

Example

Find the current in the 10Ω resistor of the circuit.



Solution:

There is only one unique node that is not in a branch. In addition there is the reference node.

1. The nodes are labeled and the voltage at the node is from the node to the reference.
2. Assume all current leaves node.
3. Write KCL at node b

$$-i_{2\Omega} - i_{10\Omega} - i_{4\Omega} = 0$$ KCL node b

4. Apply Ohm's Law to the currents.

$$i_{2\Omega} = \frac{V_b - V_a}{2} = \frac{V_b - 6}{2}$$ Ohm's Law

$$i_{10\Omega} = \frac{V_b - 0}{10}$$

$$i_{4\Omega} = \frac{V_b - V_c}{4} = \frac{V_b - 8}{4}$$

5. Solve for unknown voltage at nodes.

$$-\left(\frac{V_b - 6}{2}\right) - \left(\frac{V_b - 0}{10}\right) - \left(\frac{V_b - 8}{4}\right) = 0$$ Substitute

$$60 - 10V_b + 40 - 5V_b - 2V_b = 0$$ Solve equation

$$17V_b = 100$$
$$V_b = 5.88v$$

6. Now calculate desired value

$$i_{10\Omega} = \frac{V_b - 0}{10} = \frac{5.88v}{10\Omega} = 0.588A$$

3.6.2 Mesh - loop current

The mesh or loop method is a series approach to the problem. Mesh analysis has application where there are less loops than independent nodes.

The application of the general rules for circuit analysis yield specific steps for mesh analysis.

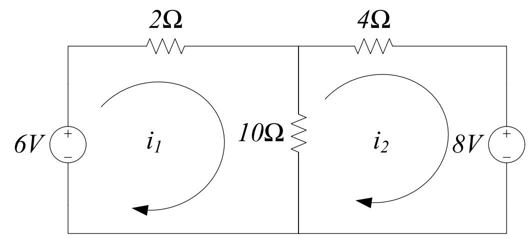
1) Assign current loops to the circuit using a "window-pane" method. Include all impedances and sources.
2) Assume the voltage polarity for the impedance elements (+ into Z)
3) Write KVL around each loop.
4) Substitute impedance elements (Ohm's law) for the voltage across the Z.
5) Solve for unknown current in the desired branch.
6) The impedances are known. Once the voltage is found at each node, then the current in each branch can be determined.

Example

Find the current in the 10Ω resistor of the circuit.

Solution:

1. Draw currents in each window pane.
2. Assume the voltage polarity for the impedance elements (+ into Z)
3. Write the KVL around each loop.



4. Apply Ohm's Law to the voltages. For illustration purposes, steps 3 and 4 are combined below.

KVL Left Loop

$$-V_{6V} + V_{2\Omega} + V_{10\Omega} = 0$$ KVL left loop

$$V_{2\Omega} = 2i_1$$ Ohm's law

$$V_{10\Omega} = 10(i_1 - i_2)$$

$$\Rightarrow -6 + 2i_1 + 10(i_1 - i_2) = 0$$ Combine

KVL Right Loop

$$V_{8V} - V_{10\Omega} + V_{4\Omega} = 0$$ KVL right loop

$$V_{4\Omega} = 4i_2$$ Ohm's law

$$V_{10\Omega} = 10(i_1 - i_2)$$

$$\Rightarrow 8 + 10(i_2 - i_1) + 4i_2 = 0$$ Combine

5. Solve for unknown current in each loop.

$$\begin{aligned} 12i_1 - 10i_2 &= 6 \\ -10i_1 + 14i_2 &= -8 \end{aligned}$$ Solve equations simultaneously

Set up using Cramer's Rule.

$$i_{1=} \frac{\begin{vmatrix} 6 & -10 \\ -8 & 14 \end{vmatrix}}{\begin{vmatrix} 12 & -10 \\ -10 & 14 \end{vmatrix}} = \frac{(6)(14) - (-10)(-8)}{(12)(14) - (-10)(-10)} = \frac{4}{68} = 0.059A$$

$$i_2 = -0.529A$$

6. Now calculate desired value

$$i_{10} = i_1 - i_2 = 0.588A$$

Cramer's Rule

- Set up a matrix for the impedance, unknowns, and RHS forcing function.
- If solving for i_n – substitute RHS for i_n in numerator.
- Use matrix of coefficients in denominator.
- Use determinants to manipulate using cross-multiplication

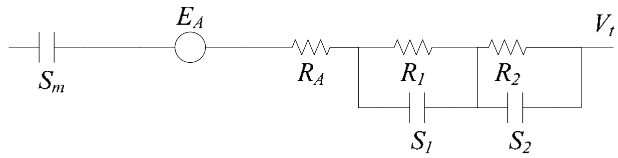
3.8 Problems

Practice Problem 1-2 (Old Style)

SITUATION:
A starting circuit is needed that will limit the starting current in a dc motor to two and one half time (2.5pu) the normal full load current.

The switches S_1 and S_2 in the starting circuit shown below are to close sequentially when the current has dropped to normal full load current (1pu)

Both switches are open when the main breaker S_M is closed.



SOLUTION:

On Closing of S_m

$$I_A = \frac{V_t}{R_1 + R_2 + R_A} = 2.5pu \Rightarrow \frac{V_t}{I_A} = R_1 + R_2 + R_A = 0.4pu$$

With V_t=1.0,

$$R_T = R_1 + R_2 + R_A = \frac{V_t}{I_A} = \frac{1}{2.5pu} = 0.4pu$$

$$V_t = E_a + I_A R_t \Rightarrow R_t = \frac{V_t - E_A}{I_A}$$

When I_A drops to 1.0 pu

$$Vt = E_a + 1(0.4pu) \Rightarrow E_A = V_t - 0.4 = 1 - 0.4 = 0.6pu$$

Close S_1 and I_A raises to 2.5pu, at that instance E_A = 0.6pu

$$R_T = R_A + R_2 = \frac{V_t - E_a}{I_A} = \frac{1-0.6}{2.5} = 0.16pu$$

When I_A again drops to 1.0 pu

$$E_A = V_t - I_A R_T = 1 - 1 * 0.16 = 0.84pu$$

Close S_2 and I_A raises to 2.5pu, at that instance E_A = 0.84pu

$$R_T = R_A = \frac{V_t - E_a}{I_A} = \frac{1-0.84}{2.5} = 0.064pu$$

When I_A again drops to 1.0 pu

$$E_A = V_t - I_A R_T = 1 - 1 * 0.064 = 0.936pu$$

$\therefore$

$$R_A = 0.064pu$$

$$R_A + R_2 = 0.16pu \Rightarrow R_2 = 0.096pu$$

$$R_A + R_2 + R_1 = 0.4pu \Rightarrow R_1 = 0.24pu$$