Chapter 11 – Laplace

11.1 Introduction

A standard waveform is defined in terms of time and frequency. The waveform contains fixed, exponential decay, and cyclic terms.

$$y(t) = F + (I - F)e^{-t/\tau}\cos(\omega t + \theta)$$

where

F = Final value (t=∞)

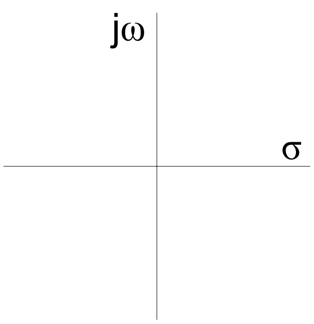
I = Initial value (t=0)

τ = time constant

A mathematical transform is often used to provide a different computational tool. The phasor representation is one transform that applies to steady state alternating circuits.

11.2 Transform

The Laplace transform is a more general technique that includes all three components. The Laplace transform is an elegant tool that converts the *d/dt* differential operator into a simple algebraic multiplier by the *s* operator. Similarly the integration operator is division by the *s* operator.



The function is transformed from time to the *s* domain, which represents a fixed and rotational component.

$$s = \sigma + j\omega$$

The definition of a Laplace depends on an exponential response operating on the time function.

$$L\left[f(t)\right] = F(s) = \int_0^\infty e^{-st} f(t)$$

The significance of the exponential operator is that it contains fixed, exponential decay and cyclic motion.

$$e^{-st} = e^{-(\sigma + j\omega)t} = e^{-\sigma t}e^{-j\omega t}$$

11.2.1 Sigma

The sigma (σ) is the coefficient of the exponential term in the time domain.

$$e^{-\sigma t} = e^{-\frac{t}{\tau}}$$

$$\sigma = \frac{1}{\tau}$$

The time constant (τ) is calculated from the impedance. The time constant depends on the damping element of resistance (R) and one of the energy storage components, electric capacitance (C) or magnetic inductance (L). In the mechanical world the energy storage would be springs or mass inertia respectively.

The time constant determines how quickly the system becomes stable. The system reaches 95% of its final value (F) in 3 time constants, 99% in 5 time constants and 99.9% in 7 time constants.

11.2.2 Omega

The omega (ω) is the frequency of the cyclic operation. The frequency is related to sinusoids.

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

The frequency (ω) is calculated from the impedance energy storage components, capacitance and inductance.

11.2.3 Differentiation

The time differentiation property of the Laplace transform will be used as the basis for investigation.

$$\frac{dN(t)}{dt} = sN(s) - I$$

Several observations can be made about the relationships. *I* is the initial condition. In a steady state, the initial condition is zero. Then the rate of time differentiation simply becomes a multiplication by the *s*-operator. Similarly, integration is simply division by the *s*-operator

$\frac{d}{dt} = s$	$\int dt = \frac{1}{s}$

11.3 Laplace Properties

Properties are definitions of mathematical computations. The mathematical manipulation of the time function and the Laplace transform follows defined rules. These properties can be individually calculated from the definition. However, they are commonly used; therefore, a tabulation of the relationships is more pragmatic.

Property	**Function**
Linear coefficient	$a\, f(t) \leftrightarrow a\, F(s)$
Linear sum	$f_1(t) + f_2(t) \leftrightarrow F_1(s) + F_2(s)$
Differention	$\frac{d\, f(t)}{dt} = f'(t) \leftrightarrow sF(s) - f(0)$
	$f''(t) \leftrightarrow s^2F(s) - sf(0) - f'(0)$
Integration	$\int_0^t f'(t)dt \leftrightarrow \frac{1}{s}F(s)$
Scaling	$f(at) \leftrightarrow \frac{1}{a}F\left(\frac{s}{a}\right)$
Frequency shift	$e^{-at}\, f(t) \leftrightarrow F(s+a)$
Initial value	$f(0) \leftrightarrow \lim_{s\to\infty} s\,F(s)$

Final value	$f(\infty) \leftrightarrow \lim_{s\to 0} s\,F(s)$
Convolution	$f_1(t) * f_2(t) \leftrightarrow F_1(s)F_2(s)$

Example

Determine the Laplace transform of the following function which represents electrical capacity.

$$i(t) = C\frac{dv(t)}{dt} + Gv(t)$$

Solution:

$$I(s) = C\left[sV(s) - v(0)\right] + Gv(s)$$

Example

Determine the Laplace transform of the following function which represents electrical capacity.

$$v(t) = \frac{1}{C}\int_0^t i(t)dt + v(0)$$

Solution:

$$V(s) = \frac{I(s)}{sC} + \frac{v(0)}{s}$$

Example

Determine the current in the following function which represents a complete circuit model.

$$v_0 u(t) = Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i(t)dt + v(0)$$

Solution:

$$\frac{V_0}{s} = RI(s) + L\left[sI(s) - i(0)\right] + \frac{I(s)}{sC} + \frac{v(0)}{s}$$

$$I(s)\left[R + sL + \frac{1}{sC}\right] = \frac{V_0}{s} + Li(0) - \frac{v(0)}{s}$$

$$I(s) = \frac{V_0 + sLi(0) - v(0)}{s^2L + sR + \frac{1}{C}}$$

$$= \frac{si(0) - \frac{v(0)}{L} + \frac{V_0}{L}}{\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}$$

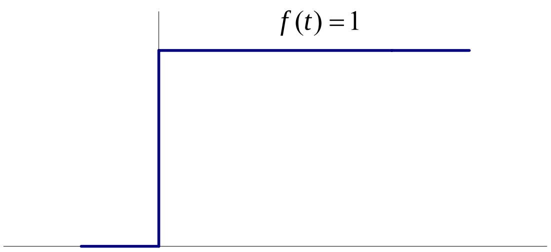
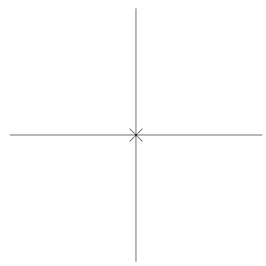
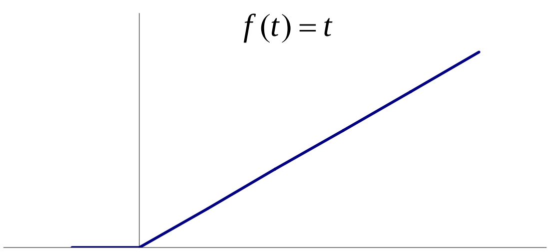
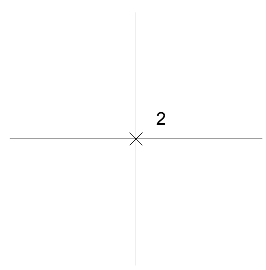
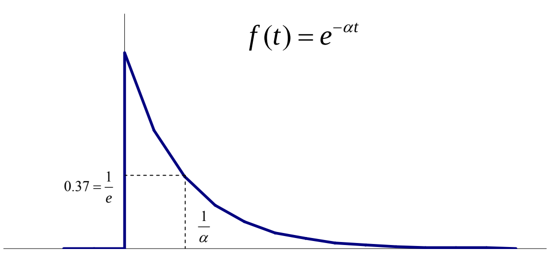
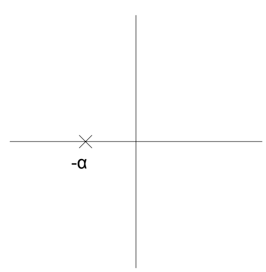
The input is a step function signal, u(t) multiplied by a magnitude, v_0. The input signal is the only value that changes for described network. In most cases, the input signal is the time function in the following transform pairs.

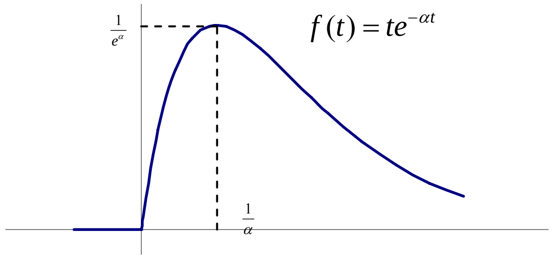
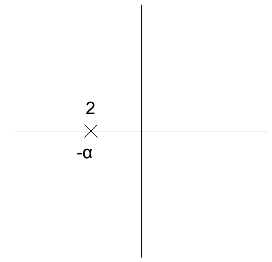
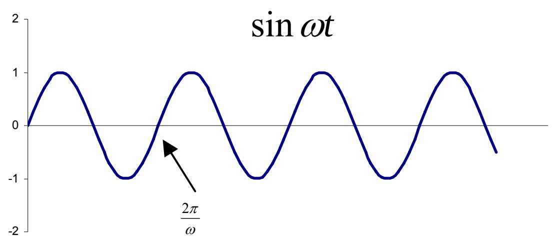
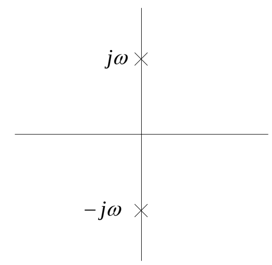
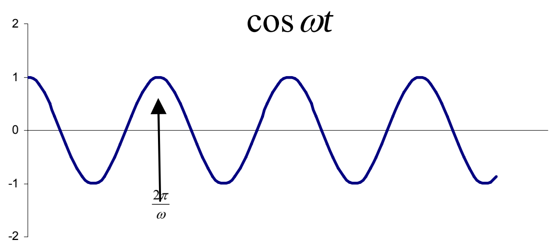
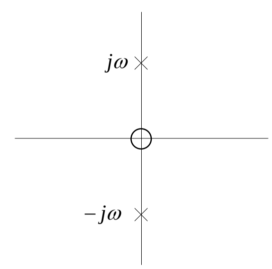
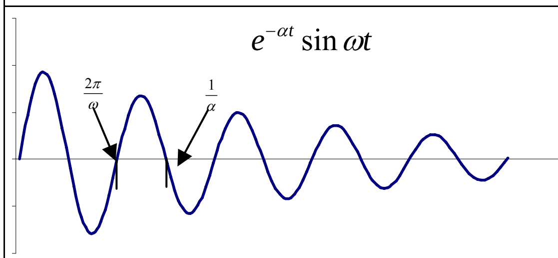
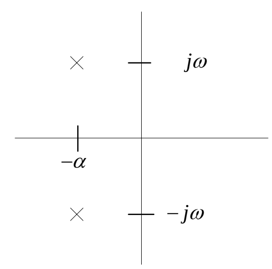
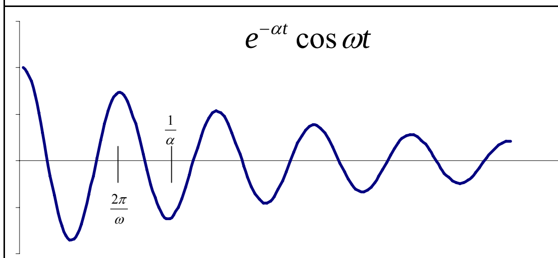
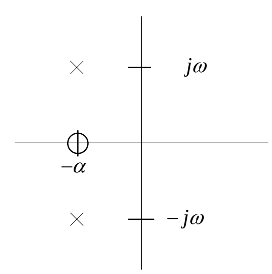
11.4 Transform pairs

Transform pairs depend on the signal into the system. The most common signals or inputs *f(t)* are given in the table. The corresponding Laplace transform *F(s)* is shown.

11.4.1 Transform pair plots

A plot of the signal is shown on the left column. The corresponding plot of the Laplace transform is shown in the right column. An X represents a denominator value called pole. An O represents a numerator value called zero.

$f(t)$	$f(t) \leftrightarrow F(s)$	$F(s)$
	$1 \leftrightarrow \frac{1}{s}$	
	$t \leftrightarrow \frac{1}{s^2}$	
	$e^{-\alpha t} \leftrightarrow \frac{1}{s+\alpha}$	

	$te^{-\alpha t} \leftrightarrow \dfrac{1}{(s+\alpha)^2}$	
	$\sin \omega t \leftrightarrow \dfrac{\omega}{s^2+\omega^2}$	
	$\cos \omega t \leftrightarrow \dfrac{s}{s^2+\omega^2}$	
	$e^{-\alpha t} \sin \omega t \leftrightarrow \dfrac{\omega}{(s+\alpha)^2+\omega^2}$	
	$e^{-\alpha t} \cos \omega t \leftrightarrow \dfrac{s+\alpha}{(s+\alpha)^2+\omega^2}$	

11.4.1 Additional transform pairs

The following transform pairs are included for convenience, but do not contain plots.

$f(t) \leftrightarrow F(s)$
$\delta(t) \leftrightarrow 1$
$t^n \leftrightarrow \frac{n!}{s^{n+1}}$
$t^n e^{-at} \leftrightarrow \frac{n!}{(s+a)^{n+1}}$
$\sin(\omega t + \theta) \leftrightarrow \frac{s\sin(\theta) + \omega\cos(\theta)}{s^2 + \omega^2}$
$\cos(\omega t + \theta) \leftrightarrow \frac{s\cos(\theta) - \omega\sin(\theta)}{s^2 + \omega^2}$

11.5 Inverse Laplace

Although the Laplace is a convenient tool for mathematical computation, the results are generally preferred in a time domain format. Therefore, the inverse of the Laplace transform is desired.

11.5.1 Partial Fraction Expansion

For rational functions it may be necessary to break the relationship into recognizable parts using partial fraction expansion. First convert the relationship into a common denominator.

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s+p_2)\ldots}$$

Then use partial fraction expansion to get individual terms for each denominator factor.

$$F(s) = \frac{k_1}{(s+p_1)} + \frac{k_2}{(s+p_2)} + \ldots$$

A variety of techniques can be used to find the numerator coefficients. The *residue* method is one. Multiply both sides of the equation by the denominator factor. Then evaluate the equation by substituting the pole (p) value in the place of s (s = -p).

$$k_i = (s+p_i)F(s)\Big|_{s=-p_i}$$ Heaviside's Theorem

$$k_1 = (s+p_1)F(s)\Big|_{s=-p_1} = \frac{N}{(-p_1+p_2)(-p_1+p_3)\ldots}$$

Once all the numerator coefficients (k) have been found, the inverse Laplace is found by looking up the term in the table.

$$f(t) = k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \ldots$$

If the denominator of the residue form had been different, then the inverse likely would have terms other than exponential decay.

11.5.2 Algebra

A relatively simple means of finding the inverse transform is to rewrite the function as a sum of terms. Each term should have a form of one of the transforms in the table.

$$F(s) = \frac{N(s)}{D(s)}$$

$$= \frac{N(s)}{(s+p_1)(s+p_2)\ldots} = \frac{k_1}{(s+p_1)} + \frac{k_2}{(s+p_2)} + \ldots$$

Multiply both sides of the equation by the denominator of the left hand side. Expand any terms on numerator of the right hand side to obtain a form with only one k_i with each term.

$$F(s) = k_1 F_1(s) + k_2 F_2(s) + \ldots$$

Equate the coefficients of the numerators. Then solve for k_i. Finally look up the inverse from the table.

$$f(t) = k_1 f_1(t) + k_2 f_2(t) + \ldots$$

11.5.3 Repeated Roots

If there are repeated roots, then the roots and descending powers of the root are separated.

$$F(s) = \frac{k_1}{(s+p_1)} + \frac{k_2}{(s+p_2)^2} + \frac{k_3}{(s+p_2)}\ldots$$

Differentiate the lower power term to determine find the coefficient.

$$k_{n-1} = \frac{d}{ds}\left[(s+p)^n F(s)\right]\Big|_{s=-p}$$

11.5.4 Complex conjugates

Sinusoids are very common signals. The result is a denominator with quadratic function. In real systems, the quadratic is a complex relationship in conjugate pairs. Normal procedures can be used, but the computation is cumbersome.

$$F(s) = \frac{k_1 s + k_2}{(s^2 + as + b)}\ldots$$

Realizing the form, the function can often be factored. This is the preferred method.

$$s^2 + as + b = (s + \alpha - j\omega) + \left(s + \alpha - j\omega\right)$$

$$= s^2 + 2\alpha s + \omega^2$$

Another approach is completing the square. In general a quadratic term can be expanded into the following form.

$$s^2 + as + b = s^2 + 2\alpha s + \alpha^2 + \omega^2$$
$$= (s+\alpha)^2 + \omega^2$$

Equate coefficients of powers to obtain the expanded terms, α and ω.

Consider the numerator. Arrange it in terms of the real offset α.

$$k_1 s + k_2 = k_1(s+\alpha) + k_3\omega$$
$$k_2 = k_1\alpha + k_3\omega$$

Now the Laplace function takes on an expanded form.

$$F(s) = \frac{k_1\left(s+\alpha\right)}{(s+\alpha)^2+\omega^2} + \frac{k_3\omega}{(s+\alpha)^2+\omega^2}$$

The inverse Laplace can be found.

$$f(t) = e^{-\alpha t}\left(k_1 \cos\omega t + k_3 \sin\omega t\right)$$

Using trigonometry, cosine and sine terms can be combined.

$$A\cos\omega t + B\sin\omega t = C\cos\left(\omega t + \theta\right)$$
$$C = \sqrt{A^2 + B^2}$$
$$\theta = \tan^{-1}\frac{B}{A}$$

Therefore, the time domain function is a cosine relationship.

$$f(t) = Ce^{-\alpha t}\cos\left(\omega t + \theta\right)$$

Example

Determine the time domain representation (Inverse Laplace) for the following function. Use residue method.

$$F(s) = \frac{12}{(s+1)(s+2)(s+3)}$$

Solution:

If the problem is not already factored, they may well be the biggest challenge. Set up the residue

$$F(s) = \frac{k_1}{(s+1)} + \frac{k_2}{(s+2)} + \frac{k_2}{(s+3)}$$

Calculate the coefficients.

$$k_1 = (s+1)F(s)\big|_{s=-1} = \frac{12}{(-1+2)(-1+3)} = 6$$

$$k_2 = (s+2)F(s)\big|_{s=-2} = \frac{12}{(-2+1)(-2+3)} = -12$$

$$k_3 = (s+3)F(s)\big|_{s=-3} = \frac{12}{(-3+1)(-3+2)} = 6$$

Write in residue form.

$$F(s) = \frac{6}{(s+1)} + \frac{-12}{(s+2)} + \frac{6}{(s+3)}$$

Take the inverse Laplace of each term.

$$f(t) = 6e^{-1t} - 12e^{-2t} + 6e^{-3t}$$

Example

Determine the time domain representation (Inverse Laplace) for the following function. Use algebra method.

$$F(s) = \frac{12}{(s+1)(s+2)}$$

$$F(s) = \frac{12}{(s+1)(s+2)} = \frac{k_1}{(s+1)} + \frac{k_2}{(s+2)}$$

$$\frac{12}{(s+1)(s+2)} = \frac{k_1\left[(s+1)(s+2)\right]}{(s+1)} + \frac{k_2\left[(s+1)(s+2)\right]}{(s+2)}$$

$$12 = k_1\left[(s+2)\right] + k_2\left[(s+1)\right]$$

$$0s = \left(k_1 + k_2\right)s \rightarrow k_1 = -k_2$$

$$12 = 2k_1 + k_2 = k_1$$

$$-12 = k_2$$

$$f(t) = 12e^{-1t} - 12e^{-2t}$$

Example

Determine the time domain representation (Inverse Laplace) for the following function with repeated roots..

$$F(s) = \frac{12}{(s+1)(s+2)^2}$$

Example

Determine the time domain representation (Inverse Laplace) for the following function with complex conjugate poles.

$$F(s) = \frac{5}{(s^2 + 4s + 25)}$$

11.6 Network equations

The purpose of studying mathematical techniques is to solve for problems in real systems.

11.6.1 Equation standard form

Equations are a symbol of how a system is constructed. There is an input, a plant, and an output.

$$in = plant \times output$$

The input is the signal that is applied. This is a known condition. The plant is how the network, system, or circuit is connected. In electrical parlance it is the impedance coefficient or RLC –"real loopy capacity" system. The output is how the plant responds to the input signal. This is the unknown values. The model can be expressed in matrix form.

$$[In] = [Coefficient][Out]$$

When investigating problems, generally the only thing that changes is the input signal. The structure of the impedance coefficient matrix and the output response stays the same. The values of the plant or coefficient matrix are determined by the physical network. The output response is how things change with a different input.

11.6.2 Correlations

Impedance is used in electrical systems to describe the system, network, or circuit. Impedance consists of three elements - resistors, capacitors, and inductors. There is a one-to-one correspondence to the mechanical elements – damper, spring, and inertial mass.

The Laplace form of these elements are shown in the table.

Function	**Relationship**	**Resistor**	**Capacitor**	**Inductor**
Impedance	$Z = \frac{V}{I}$	$Z_R = R$	$Z_L = sL$	$Z_C = \frac{1}{sC}$
Admittance	$Y = \frac{1}{Z} = \frac{I}{V}$	$Y_R = \frac{1}{R}$	$Y_L = \frac{1}{sL}$	$Y_C = sC$